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Growth of Iterated Entire Functions

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Abstract. In this paper, we study the growth of iterated entire functions of finite iterated order. Considering generalized iterations of entire functions we investigate the growth of iterated entire functions and prove some results. Our results generalize and improve some earlier results.

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1. Introduction

To investigate the growth of iterated entire functions some relevant notations and definitions are required. For standard notations and definitions we refer to [4].

Notation 1.1 [8] Let $log^{[0]}z = z$, $exp^{[0]}z = z$ and for positive integer p, $log^{[P]}z = log(log^{[p-1]}z)$, $exp^{[p]}z = exp(exp^{[p-1]}z)$.

Definition 1.2 The order $\rho_{\{f\}}$ and lower order $\lambda_{\{f\}}$ of a meromorphic function f is defined as

$$\rho_{\{f\}} = \lim_{r\to\infty} sup \frac{\log T(r,f)}{\log r}$$

and

$$\lambda_{\{f\}} = \lim_{r\to\infty} inf \frac{\log T(r,f)}{\log r}.$$

If f is an entire function, then one can easily verify that

$$\rho_{\{f\}} = \lim_{r\to\infty} sup \frac{\log^{[2]} M(r,f)}{\log r}$$

and

$$\lambda_{\{f\}} = \lim_{r\to\infty} inf \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 1.3 A function $\lambda_{\{f\}}(t)$ is called a lower proximate order of a meromorphic function f if

(i) $\lambda_{\{f\}}(t)$ is nonnegative and continuous for $t \geq t_0$, say;

(ii) $\lambda_{\{f\}}(t)$ is differentiable for $t \geq t_0$ except possibly at isolated points at which $\lambda'_{\{f\}}(t-0)$ and $\lambda'_{\{f\}}(t+0)$ exist;

(iii) $\lim_{t\to\infty} \lambda_{\{f\}}(t) = \lambda_{\{f\}} < \infty$;

(iv) $\lim_{t\to\infty} t\lambda'_{\{f\}}(t) \log t = 0$

and

(v) $\lim\, inf_{t\to\infty} \frac{T(t,f)}{t^{\lambda_{\{f\}}(t)}} = 1$.

Definition 1.4 [1] Let f and g be two entire functions defined in the open complex plane and $\tau \in (0,1]$. Then the generalized iterations of f with respect to g are defined as follows:

$$f_{\{1;g\}}(z) = (1-\tau)z + \tau f(z)$$
$$f_{\{2;g\}}(z) = (1-\tau)g_{\{1;f\}}(z) + \tau f(g_{\{1;f\}}(z))$$
$$f_{\{3;g\}}(z) = (1-\tau)g_{\{2;f\}}(z) + \tau f(g_{\{2;f\}}(z))$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$f_{\{n;g\}}(z) = (1-\tau)g_{\{n-1;f\}}(z) + \tau f(g_{\{n-1;f\}}(z))$$

and so are

$$g_{\{1;f\}}(z) = (1-\tau)z + \tau g(z)$$
$$g_{\{2;f\}}(z) = (1-\tau)f_{\{1;g\}}(z) + \tau g(f_{\{1;g\}}(z))$$
$$g_{\{3;f\}}(z) = (1-\tau)f_{\{2;g\}}(z) + \tau g(f_{\{2;g\}}(z))$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$g_{\{n;f\}}(z) = (1-\tau)f_{\{n-1;g\}}(z) + \tau g(f_{\{n-1;g\}}(z))$$

Clearly all $f_{\{n;g\}}(z)$ and $g_{\{n;f\}}(z)$ are entire functions.

Considering two transcendental entire functions f and g defined in the open complex plane $\mathbb{C}$ Clunie [2] proved that $\lim_{r\to\infty} \frac{\log T(r,f\circ g)}{T(r,f)} = \infty$ and $\lim_{r\to\infty} \frac{\log T(r,f\circ g)}{T(r,g)} = 0$. Singh [9] investigate some comparative growth properties of $\log T(r, f\circ g)$ and $T(r,f)$. Afterwards Lahiri [5] proved some results on the comparative growth of $\log T(r, f\circ g)$ and $T(r,g)$.

Recently Lahiri and Datta [6] proved some growth properties of $\log T(r, f\circ g)$ and $T(r,g)$. They [6] also investigate on the comparative growth properties of $\log\log T(r, f\circ g)$ and $T(r, f^{(k)})$.

In this paper, we study the growth of generalized iterated entire functions and prove some results which generalize and improve some earlier results.

2. Lemmas

The following lemmas will be needed in the sequel.

Lemma 2.1 [6] Let f be an meromorphic function. Then for $\theta(>0)$, the function $r^{\lambda_{\{f\}}+\theta-\lambda_{\{f\}}(r)}$ is an increasing function of r.

Lemma 2.2 [7] Let f be an entire function of finite lower order. If there exist entire functions α_i $(i = 1,2,3.\dots\dots\dots.n; n \leq \infty)$ satisfying $T(r,\alpha_i) = o(T(r,f))$ and $\sum_{i=1}^{n} \delta(\alpha_i, f) = 1$, then

$$\lim_{r\to\infty} \frac{T(r,f)}{\log M(r,f)} = \frac{1}{\pi}.$$

Lemma 2.3 [3] Let f and g be two non-constant entire functions such that $0 < \lambda_{\{f\}} \leq \rho_{\{f\}} < \infty$ and $0 < \lambda_{\{g\}} \leq \rho_{\{g\}} < \infty$. Then for any τ $(0 < \tau < \min\{\lambda_{\{f\}}, \lambda_{\{g\}}\})$

$$\log^{[n-1]} T(r, f_{\{n;g\}}) \leq \begin{cases} (\rho_{\{f\}} + \tau)(1 + O(1)) \log M(r,g) + O(1), & when\ n\ is\ even \\ (\rho_{\{g\}} + \tau)(1 + O(1)) \log M(r,f) + O(1), & when\ n\ is\ odd \end{cases}$$

and

$$\log^{[n-1]} T(r, f_{\{n;g\}}) \geq \begin{cases} (\lambda_{\{f\}} - \tau)(1 + O(1)) \log M\left(\frac{r}{4^{n-1}}, g\right) + O(1), & when\ n\ is\ even \\ (\lambda_{\{g\}} - \tau)(1 + O(1)) \log M\left(\frac{r}{4^{n-1}}, f\right) + O(1), & when\ n\ is\ odd, \end{cases}$$

for all large values of r.

3. Theorems

Theorem 3.1 Let f and g be two non-constant entire functions such that $\lambda_{\{g\}}$ and $\lambda_{\{f\}}(>0)$ are finite. If there exist entire functions α_i $(i = 1,2,3,\dots\dots\dots\dots, n; n \leq \infty)$ satisfying $T(r,\alpha_i) = o(T(r,f))$ as $r \to \infty$ and $\sum_{i=1}^{n} \delta(\alpha_i, f) = 1$, then

$$\frac{\pi\lambda_{\{g\}}}{(4^{n-1})^{\lambda_{\{f\}}}} \leq \limsup_{r\to\infty} \frac{\log^{[n-1]} T\big(r, f_{\{n;g\}}\big)}{T(r,f)} \leq \pi\rho_{\{g\}},\ \ when\ n\ is\ odd.$$

Proof. If $\lambda_{\{g\}} = 0$ then the first inequality is obvious. Now we suppose that $\lambda_{\{g\}} > 0$. For $0 < \tau < \min\{1,\ \lambda_{\{f\}},\ \lambda_{\{g\}}\}$ and odd n we have from Lemma 2.3, for all large values of r

$$\frac{\log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T(r,f)}$$

$$\geq \left(1+O(1)\right)\left(\lambda_{\{g\}}-\tau\right)\frac{\log M\left(\frac{r}{4^{n-1}}, f\right)}{T(r,f)} + O(1)$$

$$\geq \left(1+O(1)\right)\left(\lambda_{\{g\}}-\tau\right)\frac{\log M\left(\frac{r}{4^{n-1}}, f\right)}{T\left(\frac{r}{4^{n-1}}, f\right)}\frac{T\left(\frac{r}{4^{n-1}}, f\right)}{T(r,f)}$$

$$+ O(1) \qquad (1)$$

Let $0 < \tau < \min\{1,\ \lambda_{\{f\}},\ \lambda_{\{g\}}\}$.
Since

$$\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_{\{f\}}(r)}} = 1,$$

there exists a sequence of values of r tending to infinity for which

$$T(r,f) < (1+\tau) r^{\lambda_{\{f\}}(r)} \qquad (2)$$

and

$$T(r,f) > (1-\tau) r^{\lambda_{\{f\}}(r)} \qquad (3)$$

Hence from (2) and (3) we have for $\delta(>0)$ and for a sequence of values of r tending to infinity

$$\frac{T(\frac{r}{4^{n-1}}, f)}{T(r,f)} > \left(1+O(1)\right)\frac{(1-\tau)}{(1+\tau)}\frac{(\frac{r}{4^{n-1}})^{\lambda_{\{f\}}+\delta}}{(\frac{r}{4^{n-1}})^{\lambda_{\{f\}}+\delta-\lambda_{\{f\}}(\frac{r}{4^{n-1}})}}\frac{1}{(r)^{\lambda_{\{f\}}(r)}}$$

$$\geq \left(1+O(1)\right)\frac{(1-\tau)}{(1+\tau)}\frac{1}{(4^{n-1})^{\lambda_{\{f\}}+\delta}},$$

since by Lemma 2.1, $(r)^{\lambda_{\{f\}}+\delta-\lambda_{\{f\}}(r)}$ is an increasing function of r.
Since $\tau,\ \delta > 0$ are arbitrary, therefore using Lemma 2.2 we have from (1)

$$\limsup_{r\to\infty} \frac{\log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T(r,f)} \geq \frac{\pi\lambda_{\{g\}}}{(4^{n-1})^{\lambda_{\{f\}}}}.$$

If $\rho_{[g]} = \infty$, the second inequality is obvious. So we may assume $\rho_{[g]} < \infty$. Then the second inequality follows from Lemma 2.2 and Lemma 2.3.
This proves the theorem.

Theorem 3.2 Let f and g be two non-constant entire functions such that $\rho_{\{g\}} < \lambda_{\{f\}} \leq \rho_{\{f\}} < \infty$. Then for $k = 0, 1, 2, 3, \dots \dots$

$$\lim_{r\to\infty} \frac{log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T(r, f^{(k)})} = 0, when\ n\ is\ even.$$

Proof. Let n be an even integer. Then from Lemma 2.3 we have for arbitrary $\tau\ (>0)$ such that $\rho_{\{g\}} + \tau < \lambda_{\{f\}} - \tau$ and for large values of r,

$$\log^{[n-1]} T\left(r, f_{\{n;g\}}\right) \leq \left(\rho_{\{f\}} + \tau\right)\left(1 + O(1)\right) \log M(r, g) + O(1),$$

$$\log M(r, g) < r^{\rho_{\{g\}}+\tau}$$

and $\quad T(\mathrm{r}, f^{(k)}) > r^{(\lambda_{\{f\}}-\tau)}$

Therefore

$$\frac{\log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T(\mathrm{r}, f^{(k)})} \leq \frac{(\rho_{\{f\}} + \tau) r^{(\rho_{\{g\}}+\tau)}}{r^{(\lambda_{\{f\}}-\tau)}} + o(1)$$

Hence

$$\lim_{r\to\infty} \frac{\log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T(\mathrm{r}, f^{(k)})} = 0.$$

This proves the theorem.

Theorem 3.3 Let f and g be two non-constant entire functions such that $\rho_{\{f\}} < \lambda_{\{g\}} \leq \rho_{\{g\}} < \infty$. Then for $k = 0, 1, 2, 3, \dots \dots$

$$\lim_{r\to\infty} \frac{log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T(r, g^{(k)})} = 0, when\ n\ is\ odd.$$

Proof. Let n be an odd integer. Then from Lemma 2.3 we have for arbitrary $\tau\ (> 0)$ such that $\rho_{\{f\}} + \tau < \lambda_{\{g\}} - \tau$ and for large values of r,

$$\log^{[n-1]} T\left(r, f_{\{n;g\}}\right) \leq \left(\rho_{\{g\}} + \tau\right)\left(1 + O(1)\right) \log M(r, f) + O(1),$$

$$\log M(r, f) < r^{\rho_{\{f\}}+\tau}$$

and $\quad T(\mathrm{r}, g^{(k)}) > r^{(\lambda_{\{g\}}-\tau)}$

Therefore

$$\frac{\log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T(\mathrm{r}, g^{(k)})} \leq \frac{(\rho_{\{g\}} + \tau) r^{(\rho_{\{f\}}+\tau)}}{r^{(\lambda_{\{g\}}-\tau)}} + o(1)$$

Hence

$$\lim_{r\to\infty} \frac{\log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T(\mathrm{r}, g^{(k)})} = 0.$$

This proves the theorem.

Theorem 3.4 Let $f\ and\ g$ be two non-constant entire functions such that $0 < \lambda_{\{g\}} \leq \rho_{\{g\}} < \infty$ and $\rho_{\{f\}} < \infty$. *Then for* $k = 0, 1, 2, 3, \dots \dots$

$$\lim_{r\to\infty} \frac{log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T\left(exp(r), g^{(k)}\right)} = 0,$$

for all natural numbers $n (\geq 2)$.

Proof. Let us first suppose that n is even. Then for all sufficiently large values of r and τ $(0 < \tau < \lambda_{\{g\}})$ we have by Lemma 2.3

$$\log^{[n-1]} T\left(r, f_{\{n;g\}}\right) \leq (\rho_{\{f\}} + \tau)(1 + O(1)) \log M(r, g) + O(1),$$

$$\log M(r, g) < r^{\rho_{\{g\}}+\tau}$$

and $\quad T(\exp(r), g^{(k)}) > e^{r^{(\lambda_{\{g\}}-\tau)}}$

So

$$\frac{\log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T(\exp(r), g^{(k)})} \leq \frac{(\rho_{\{f\}} + \tau) r^{(\rho_{\{g\}}+\tau)}}{e^{r^{(\lambda_{\{g\}}-\tau)}}} + o(1)$$

Therefore

$$\lim_{r\to\infty} \frac{\log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T(\exp(r), g^{(k)})} = 0.$$

Now let n is odd. Then by Lemma 2.3 we have

$$\log^{[n-1]} T\left(r, f_{\{n;g\}}\right) \leq (\rho_{\{g\}} + \tau)(1 + O(1)) \log M(r, f) + O(1),$$

$$\log M(r, f) < r^{(\rho_{\{f\}}+\tau)}$$

So

$$\frac{\log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T(\exp(r), g^{(k)})} \leq \frac{(\rho_{\{g\}} + \tau) r^{(\rho_{\{f\}}+\tau)}}{e^{r^{(\lambda_{\{g\}}-\tau)}}} + o(1).$$

Therefore

$$\lim_{r\to\infty} \frac{\log^{[n-1]} T\left(r, f_{\{n;g\}}\right)}{T(\exp(r), g^{(k)})} = 0.$$

This proves the theorem.

4. CONCLUSION

In this paper, we investigate the growth properties of iterated entire functions. Considering generalized iterations of entire functions of finite iterated order we prove some growth properties of generalized iterated entire functions.

As the results obtained are quite general in nature we can reduce the general results to the corresponding special results by assigning different values to the parameters involved in the general results. Thus these results can be applied to different branches of science and engineering especially in mathematics.

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